Dynamic Systems: Eigenvalues, Eigenvectors and the Jordan Decomposition

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Sources
Stokey and Lucas with Prescott, 1989
Simon and Blume, 1994

1 Introduction

• The objective of many exercises in macroeconomics is to find a stable solution to a dynamic system.
• Usually, the dynamic system that one works with is the result of (first) optimization via dynamic programming or optimal control and (later) linearization around a steady state. The linearized system is usually a dynamic system whose solution displays the behavior of the model.
• Thus, we are usually looking for a solution that is stable.
• Uniqueness would be nice but that may be too much to ask for.
• Usually, the multiplicity of solutions is narrowed down (somewhat) by applying the rational expectations hypothesis.

2 Eigenvalues and Eigenvectors

The **eigenvalues** of a square matrix A are given by the roots of the characteristic equation:

$$det(A - \lambda I) = 0$$

• Subtracting eigenvalues from the diagonal elements makes the resulting matrix a singular matrix.
• Eigenvalues can be real and distinct, real and repeated or complex and distinct or complex and repeated.
• Real and distinct eigenvalues are the easiest to handle
• Repeated eigenvalues provide unique challenges to the solution of the dynamic system (later sections)
• For discrete systems:
∘ If A has k linearly independent eigenvalues (so that $det[A - \lambda I]$ is non-singular), then the diagonalized matrix produces k equations, each of which depends on its respective eigenvalues. Since the eigenvalues are independent, the solutions will be independent as well

∘ If the eigenvalues are not linearly independent, then the solutions may not be independent of each other so that we have fewer than k independent solutions for k equations. The steps to be taken in that case are described in a section entitled "Repeated Eigenvalues"

∘ Eigenvalues that are linearly independent have a multiplicity of one. Eigenvalues that are repeated m times have multiplicity m

∘ If all eigenvalues have a absolute value less than unity, then the solution is stable for all initial values of the system

∘ If some (or all) eigenvalues have an absolute value greater than or equal to unity, the solution may be unstable for some initial values of the system

∘ If some of the eigenvalues have complex roots, then the eigenvalues and eigenvectors come in conjugate pairs. The complex conjugate pairs will have a real part and an imaginary part. The imaginary part affects the oscillations of the system so we are less interested in it (even though it makes for a nice graphical presentation. The real part of the eigenvalue is relevant for stability and follows the same rules/restrictions as before.

∘ To clarify (?) matters further, some eigenvalues could have complex that have multiplicity greater than unity but the system would be stable as long as the real parts of the complex, repeated roots are less than one in absolute value.

Some helpful observations:
• The eigenvalues of a diagonal matrix are the diagonal elements
• A matrix is singular iff 0 is an eigenvalue

The **eigenvectors** of a square matrix A are non-zero vectors v such that:

$$\begin{aligned}(A - \lambda I)v &= 0 \\ \Rightarrow Av &= \lambda v\end{aligned}$$

• Since $(A - \lambda I) = 0$ for sure only if v is non-zero.

2.1 Operationally Speaking

• First find the eigenvalues by solving the characteristic equation given by the determinant of $(A - \lambda I)$
• Then plug in solutions to λ one-by-one into $(A - \lambda I)v = 0$ so that the only unknown now is v.

• For each λ, obtain an eigenvector equal in size to the number of columns (or equivalently) rows of A.
• All multiples of one eigenvector A will also be eigenvectors of A
• There can be multiple (linearly independent) eigenvectors for *each* of the eigenvalues
• The simplest eigenvector is usually considered
• If there are h distinct eigenvalues, then it is always possible to find h linearly independent eigenvectors.

3 Linear Difference Equations

Write the general system non-homogeneous linear difference equations in the form:

$$x_{t+1} = Ax_t + a_0 \tag{1}$$

where A is the coefficient matrix so that the unique stationary point (if A is non-singular) is given by

$$\bar{x} = (I - A)^{-1} a_0 \tag{2}$$

If $z_t = x_t - \bar{x}$, then these deviations must satisfy

$$z_{t+1} = Az_t \tag{3}$$

The solution to this problem is obviously $z_t = A^t z_0$ which implies that we need to know more about the behavior of A^t.

3.1 Using the Jordan Decomposition

• The objective here is to convert A into a diagonal matrix so that solving the system is easy and along the lines of the solution to the single dimensional system
• Geometrically, this is a change of coordinates. Algebraically, this is done via an appropriate transformation matrix. The appropriate transformation is given by the Jordan decomposition $A = B^{-1}\Lambda B$, where Λ is the Jordan matrix.
• The matrix B is non-singular and the Jordan matrix is a block diagonal matrix given by

$\Lambda = diag[\Lambda_1, \ldots, \Lambda_k]$. Each block has λ_i on the diagonal and 1 as the element to the right of the diagonal.
• The Jordan transformation matrix and the resulting transformation (i.e., the diagonalized matrix A) are intrinsically linked to eigenvectors and eigenvalues (respectively). To see this, write the Jordan decomposition in the following form:

$$\begin{aligned} A &= B^{-1}\Lambda B \\ \Rightarrow BA &= \Lambda B \\ \Rightarrow BA &= \lambda I B \\ \Rightarrow B(A - \lambda I) &= 0 \end{aligned}$$

It is clear that the Jordan matrix contains the eigenvalues (λ_i) of A. Similarly, B satisfies the definition of the eigenvectors of A.
• If A has n distinct characteristic roots then the Jordan matrix is diagonal and the diagonal elements are the eigenvalues of A.

3.2 Applying Jordan methods

If we define a new variable:

$$w_t = Bz_t = B(x_t - \bar{x}) \tag{4}$$

From equation (3), this implies that

$$\begin{aligned} w_{t+1} &= Bz_{t+1} \\ &= BAz_t \\ &= B[B^{-1}\Lambda B]z_t \\ &= \Lambda B z_t \\ &= \Lambda w_t \end{aligned}$$

so that the solution can be written as

$$w_t = \Lambda^t w_0 \tag{5}$$

• Obviously, the system is easy to solve if the Λ matrix is diagonal (as opposed to block-diagonal).

We still have to recover the solution of the system in terms of the original variable x. This is easy since equation (4) and equation (5) provide us with

$$\begin{aligned} w_t &= \Lambda^t w_0 \\ Bz_t &= \Lambda^t B z_0 \end{aligned}$$

so that the solution in terms of z is given by:

$$z_t = B^{-1}\Lambda^t B z_0 \tag{6}$$

Using equation (3), the general solution of the system in terms of the original variable is:

$$x_t = \bar{x} + B^{-1}\Lambda^t B(x_0 - \bar{x}) \tag{7}$$

where x_t is a $k \times 1$ vector and the the value of $\bar{x}$ is given by equation (2)
• The general solution above can also be written in the following form:

$$\begin{aligned} x_t &= [I - B^{-1}\Lambda^t B]\bar{x} + B^{-1}\Lambda^t B x_0 \\ &= [I - B^{-1}\Lambda^t B][I - A]^{-1}a_0 + B^{-1}\Lambda^t B x_0 \\ &= [I - A^t][I - A]^{-1}a_0 + A^t x_0 \end{aligned}$$

where the second equation simply replaces $\bar{x}$ by its value from equation (2). Note that the first term in the last equation now is independent of x_0. This form of the solution adds nothing new, it simply writes out the solution in a familiar form - separating the effects of a_0 and x_0.
• To get explicit solutions, we need (i) functional form for x, (ii) initial values: x_0 and (iii) values for a_0

3.3 Element-by-Element Solution

• If the eigenvalues are linearly independent (so that the Jordan matrix is diagonal), then solutions for individual elements of $x_t = (x_t^1, \ldots, x_t^k)$ are given by the following calculations

• Equation (5) gives the solution for the $k \times 1$ system in terms of a $k \times k$ diagonal matrix Λ^t and and $k \times 1$ vector of initial conditions. A typical element in row h on the left-hand side is then directly equated to its corresponding element on the right hand side:

$$w_t^h = \lambda^t w_0$$

• From equation (3) and the resulting equation (6), we have $z_t = B^{-1}\Lambda^t B z_0$ so that a typical element in row i of the $k \times 1$ vector z_t is given by:

$$z_t^i = [u_i \cdot \lambda_i^t \cdot v_i] z_0^i$$

where the vector u_i is the i^{th} column of B^{-1}. Note that the expression in brackets is a dot product so it is really a sum. To keep the notation from becoming cumbersome, we allow the index variable i to appear as a subscript for eigenvalues and eigenvectors (since they have inverses and exponents as superscripts).
• Finally, using equation (3) puts the solution in terms of x:

$$x_t^i = \bar{x} + [u_i \cdot \lambda^t \cdot v_i](x_0^i - \bar{x})$$

3.4 Reconciling Notation with Simon and Blume

• First, note that SB start with a homogeneous system of equations so that their starting point is $z_{t+1} = Az_t$ rather than the non-homogenous system $x_{t+1} = Ax_t + a_0$. This implies that we have to remember to go one step further than the solution in SB to get an expression for the non-homogeneous system.
• SB use the transformation $z = PZ$ which is equivalent to equation (4) here.
• Finally, the equivalence between the two solutions is made clear when we realize that the definition of the vector $c = [c_1, \ldots, c_k]'$ in SB is the same as Bz_0 here so that c includes terms u_i as well as z_0.